SOAL-SOAL OLIMPIADE MATEMATIKA DAN PENYELESAIANNYA

1. Buktikan untuk setiap bilangan real a, b berlaku $a^2 + b^2 \geq 2ab$!

   Bukti :
   $(a - b)^2 \geq 0 \Leftrightarrow a^2 - 2ab + b^2 \geq 0 \Leftrightarrow a^2 + b^2 \geq 2ab$

2. Buktikan untuk setiap bilangan real a, b dengan $a \geq 0 \text{ dan } b \geq 0$ berlaku $\frac{a+b}{2} \geq \sqrt{ab}$!

   Bukti :
   $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 \Leftrightarrow a - 2\sqrt{ab} + b \geq 0 \Leftrightarrow a + b \geq 2\sqrt{ab} \Leftrightarrow \frac{a+b}{2} \geq \sqrt{ab}$

Catatan : Bentuk $\frac{a+b}{2} \geq \sqrt{ab}$ dikenal sebagai AM ≥ GM dimana AM singkatan Arithmetic Mean sedangkan GM singkatan Geometric Mean.

3. Buktikan untuk setiap bilangan positif a, b, c dan d berlaku $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$!

   Bukti :
   $$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt{abcd}$$

4. Buktikan untuk setiap bilangan real a, b dan c dengan $a \geq 0, b \geq 0 \text{ dan } c \geq 0$ berlaku $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

   Bukti :
   Misal $\frac{a+b+c}{3} = x \Leftrightarrow a + b + c = 3x \quad dan \quad \sqrt[3]{abc} = y \Leftrightarrow abc = y^3$

   Maka $\frac{a+b+c+x}{4} = \frac{\frac{a+b}{2} + \frac{c+x}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+x}{2}\right)} \geq \sqrt{\sqrt{ab}\sqrt{cx}} = \sqrt[4]{abcx} = \sqrt[4]{y^3x}$

   Karena $a + b + c = 3x$ maka $\frac{3x + x}{4} \geq \sqrt[4]{y^3x} \Leftrightarrow x^4 \geq y^3x \Leftrightarrow x \geq y$

5. Buktikan untuk setiap bilangan positif a, b, c berlaku $(b + c)(c + a)(a + b) \geq 8abc$!

   Bukti :

   $\frac{b+c}{2} \geq \sqrt{bc}$ ....(1)

   $\frac{c+a}{2} \geq \sqrt{ca}$ ....(2)

   $\frac{a+b}{2} \geq \sqrt{ab}$ ....(3)

   Jika (1) x (2) x (3) maka didapat : $\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right)\left(\frac{a+b}{2}\right) \geq \sqrt{a^2b^2c^2} = abc$

   Atau $(b + c)(c + a)(a + b) \geq 8abc$

6. Jika a bilangan positif, buktikan bahwa $a + \frac{1}{a} \geq 0$!

Bukti :

$$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0 \Leftrightarrow a - 2 + \frac{1}{a} \geq 0 \Leftrightarrow a + \frac{1}{a} \geq 2$$

7. Jika a dan b sembarang bilangan, buktikan bahwa $\frac{a}{b} + \frac{b}{a} \geq 2$!

Bukti :

$$(a - b)^2 \geq 0 \Leftrightarrow a^2 + b^2 \geq 2ab \Leftrightarrow \frac{a}{b} + \frac{b}{a} \geq 2$$

8. Jika a, b bilangan positif dan a + b = 1 maka $ab \leq \frac{1}{2}$!

Bukti :

Karena a dan b positif dan a + b = 1 maka :

$$\frac{1}{a} \geq 1 \quad ....(1)$$

$$\frac{1}{b} \geq 1 \quad ....(2)$$

Jika (1) + (2) maka $\frac{1}{a} + \frac{1}{b} \geq 2 \Leftrightarrow \frac{a+b}{ab} \geq 2 \Leftrightarrow a + b \geq 2ab \Leftrightarrow 1 \geq 2ab \Leftrightarrow ab \leq \frac{1}{2}$

9. Jika a, b, c , d bilangan positif, maka buktikan $(ac + bd)(ab + cd) \geq 4abcd$!

Bukti :

$$\frac{a}{b} + \frac{b}{a} \geq 2 \quad ....(1) \quad dan \quad \frac{c}{d} + \frac{d}{c} \geq 2 \quad ....(2)$$

Jika (1) + (2) didapat :

$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \geq 4 \quad \Leftrightarrow \quad \frac{a}{b} + \frac{d}{c} + \frac{c}{d} + \frac{b}{a} \geq 4$$

$$\frac{a^2cd + abd^2 + abc^2 + b^2cd}{abcd} \geq 4 \quad \Leftrightarrow \quad (ac + bd)(ab + cd) \geq 4abcd$$

10. Untuk setiap bilangan real x, buktikan bahwa $\frac{x^2}{1 + x^4} \leq \frac{1}{2}$!

Bukti :

$$(x^2 - 1)^2 \geq 0 \quad \Leftrightarrow \quad x^4 - 2x^2 + 1 \geq 0 \quad \Leftrightarrow \quad x^4 + 1 \geq 2x^2 \quad \Leftrightarrow \quad \frac{x^2}{1 + x^4} \geq 2$$

11. Untuk setiap bilangan real x, buktikan bahwa $\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2$!

Bukti :

$$x^4 \geq 0 \quad \Leftrightarrow \quad x^4 + 4x^2 + 4 \geq 4x^2 + 4 \quad \Leftrightarrow \quad (x^2 + 2)^2 \geq 4(x^2 + 1)$$

$$\Leftrightarrow \quad (x^2 + 2)^2 \geq \left(2\sqrt{x^2 + 1}\right)^2 \quad \Leftrightarrow \quad x^2 + 2 \geq 2\sqrt{x^2 + 1} \quad \Leftrightarrow \quad \frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2$$

12. Hitunglah nilai dari :

$$\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \sqrt{1 + \frac{1}{3^2} + \frac{1}{4^2}} + ... + \sqrt{1 + \frac{1}{2004^2} + \frac{1}{2005^2}}$$

Jawab :

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} = \sqrt{\frac{n^2(n+1)^2 + (n+1)^2 + n^2}{n^2(n+1)^2}} = \sqrt{\frac{n^2(n^2 + 2n + 1) + (n^2 + 2n + 1) + n^2}{(n(n+1))^2}}$$

$$= \sqrt{\frac{n^4 + 2n^3 + 3n^2 + 2n + 1}{(n(n+1))^2}} = \sqrt{\frac{(n^2 + n + 1)^2}{(n(n+1))^2}} = \frac{n^2 + n + 1}{n^2 + n} = 1 + \frac{1}{n^2 + n} = 1 + \frac{1}{n(n+1)}$$

$$= 1 + \frac{1}{n} - \frac{1}{n+1}$$

Jadi $\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \sqrt{1 + \frac{1}{3^2} + \frac{1}{4^2}} + ... + \sqrt{1 + \frac{1}{2004^2} + \frac{1}{2005^2}}$

$$= \left(1 + \frac{1}{1} - \frac{1}{2}\right) + \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \left(1 + \frac{1}{3} - \frac{1}{4}\right) + ... + \left(1 + \frac{1}{2004} - \frac{1}{2005}\right)$$

$$= (1 + 1 + 1 + ... + 1) + \left(1 - \frac{1}{2005}\right) = 2004 + \frac{2004}{2005} = 2004\frac{2004}{2005}$$

13. Diketahuia, b, c, d dan e adalah bilangan real. Jika a+b+c+d+e= 19 dan $a^2 + b^2 + c^2 + d^2 + e^2 = 99$ tentukan nilai maksimumz !

<u>Jawab</u> :

$(19 - e)^2 = (a + b + c + d)^2$

$361 - 38e + e^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$

$361 - 38e + e^2 = 99 - e^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$

$361 - 38e + e^2 \le 99 - e^2 + a^2 + b^2 + a^2 + c^2 + a^2 + d^2 + b^2 + c^2 + b^2 + d^2 + c^2 + d^2$

$361 - 38e + e^2 \le 99 - e^2 + 99 - e^2 + 99 - e^2 + 99 - e^2$

$361 - 38e + e^2 \le 396 - 4e^2$

$5e^2 - 38e - 35 \le 0$

Dengan rumus abc didapat $\frac{38 - \sqrt{2144}}{10} \le e \le \frac{38 + \sqrt{2144}}{10}$

Jadi nilai maksimume $= \frac{38 + \sqrt{2144}}{10}$

14. Jika 1+2+3+4+….+n=aaa, maka tentukan nilai n dan aaa !

<u>Jawab</u> :

$1 + 2 + 3 + ... + n = \frac{n}{2}(n+1)$

$aaa = 111xa = (3xa)x37$

$\frac{n}{2}(n+1) = (3xa)x37$

$n(n+1) = (6xa)x37$

Ini merupakan perkalian berurutan.

Jadi a = 6 dan n = 36

15. Jika aabb = $(xy)^2$ maka tentukan nilai dari a, b, x dan y !

<u>Jawab</u> :

Karena $(xy)^2$ adalah bilangan kuadrat maka angka satuannya 0, 1, 4, 5, 6 atau 9.

Berarti bb = 00, 11, 44, 55, 66 atau 99

Bilangan kuadrat bila dibagi 4 sisanya 0 (untuk genap) atau 1 (untuk ganjil)

Bilangan habis dibagi 4 jika 2 angka terakhir habis dibagi 4, jadi bb = 44

aabb = aa44 = 11 x a04 maka a = 7

aabb = 7744 = 88^2

Sehingga a = 7, b = 4, x = 8 dan y = 8

16. Buktikan bahwa : $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2 = \left[\frac{1}{2}n(n+1)\right]^2$

Bukti :

Dibuktikan dengan induksi matematika.

Untuk n = 1 maka $1^3 = \left[\frac{1}{2}.1(1+1)\right]^2$ benar

Misal untuk n = k benar maka $1^3 + 2^3 + 3^3 + ... + k^3 = \left[\frac{1}{2}k(k+1)\right]^2$

Untuk n = k + 1 maka $1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$

$$= \left[\frac{1}{2}k(k+1)\right]^2 + (k+1)^3$$

$$= (k+1)^2\left(\frac{1}{4}k^2 + k + 1\right)$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \left[\frac{1}{2}(k+1)(k+2)\right]^2$$

17. Jika $2004^3 = A^2 - B^2$ dimana A dan B bilangan asli, maka tentukan nilai A dan B !

Jawab :

$$1^3 + 2^3 + 3^3 + ... + (n-1)^3 + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$$

$$\left[\frac{1}{2}(n-1)n\right]^2 + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$$

$$n^3 = \left[\frac{1}{2}n(n+1)\right]^2 - \left[\frac{1}{2}(n-1)n\right]^2$$

$$2004^3 = \left[\frac{1}{2}.2004.2005\right]^2 - \left[\frac{1}{2}.2003.2004\right]^2$$

$$= (1002.2005)^2 - (1002.2003)^2$$

Jadi A = 1002.2005 dan B = 1002.2003

18. Jika $A = 1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + ... + 2005^3$, maka tentukan nilai A !

Jawab :

$$\left(1^3 + 2^3 + 3^3 + ... + 2005^3\right) - 2(2^3 + 3^3 + ... + 2004^3)$$

$$= \left(1^3 + 2^3 + 3^3 + ... + 2005^3\right) - 2.2^3(1^3 + 2^3 + 3^3 + ... + 1002^3)$$

$$= \left(\frac{1}{2}.2005.2006\right)^2 - 16(\frac{1}{2}.1002.1003)^2$$

$$= 1003^2(2005^2 - (4.501)^2)$$

$$= 1003^2(2005^2 - 2004^2)$$

$$= 1003^2(2005 + 2004)(2005 - 2004)$$

$$= 1003^2.4009$$

19. $a_1, a_2, a_3, ..., a_n$ adalah bilangan cacah yang berbeda. Jika $2^{a_1} + 2^{a_2} + 2^{a_3} + ... + 2^{a_n} = 2005$ maka tentukan nilai dari $a_1 + a_2 + a_3 + ... + a_n$!

Jawab :

$2005 = 11111010101_2$

$2005 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 0 + 2^4 + 0 + 2^2 + 0 + 2^0$

$a_1 + a_2 + a_3 + ... + a_n = 10 + 9 + 8 + 7 + 6 + 4 + 2 + 0 = 46$

20. Diketahui x, y, z dan t adalah bilangan real yang tidak nol dan memenuhi persamaan :

$x + y + z = t$ .... (1)

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{t}$ .... (2)

$x^3 + y^3 + z^3 = 1000^3$ .... (3)

Tentukan nilai dari x + y + z + t

Jawab :

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy + xz + yz}{xyz} = \frac{1}{t} \Leftrightarrow \quad xy + xz + yz = \frac{xyz}{t}$$

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3(x + y + z)(xy + xz + yz) - 3xyz$$

$$t^3 = x^3 + y^3 + z^3 + 3t.\frac{xyz}{t} - 3xyz$$

$$x^3 + y^3 + z^3 = t^3 = 1000^3 \rightarrow t = 1000$$

$$x + y + z + t = t + t = 2t = 2000$$

21. Suatu fungsi dinyatakan sebagai $f(x) = \frac{e^x}{e^x + \sqrt{e}}$.

Tentukan nilai dari $f(\frac{1}{2005}) + f(\frac{2}{2005}) + ... + f(\frac{2004}{2005})$

Jawab :

$$f(x) + f(1-x) = \frac{e^x}{e^x + \sqrt{e}} + \frac{e^{1-x}}{e^{1-x} + \sqrt{e}} = \frac{e + e^x\sqrt{e} + e + e^{1-x}\sqrt{e}}{e^x e^{1-x} + e^x\sqrt{e} + e^{1-x}\sqrt{e} + e} = \frac{2e + e^x\sqrt{e} + e^{1-x}\sqrt{e}}{2e + e^x\sqrt{e} + e^{1-x}\sqrt{e}} = 1$$

$$f(\frac{1}{2005}) + f(\frac{2004}{2005}) = 1$$

$$f(\frac{2}{2005}) + f(\frac{2003}{2005}) = 1$$

.....

.....

$$f(\frac{1002}{2005}) + f(\frac{1003}{2005}) = 1$$

____________________ +

= 1002

22. Diketahui a dan b adalah bilangan real yang memenuhi syarat :

i. $a^3 - 3ab^2 = 44$

ii. $b^3 - 3a^2b = 8$

Tentukan nilai $a^2 + b^2$!

Jawab :

$a^3 - 3ab^2 = 44 \Rightarrow (a^3 - 3ab^2)^2 = 44^2 \Leftrightarrow a^6 - 6a^4b^2 + 9a^2b^4 = 1936$

$b^3 - 3a^2b = 8 \Rightarrow (b^3 - 3a^2b)^2 = 8^2 \Leftrightarrow b^6 - 6a^2b^4 + 9a^4b^2 = 64$ +

$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = 2000$

$(a^2 + b^2)^3 = 2000 \Leftrightarrow a^2 + b^2 = \sqrt[3]{2000} = 10\sqrt[3]{2}$

23. Tentukan banyaknya bilangan yang terdiri dari 4 digit angka abcd sehingga memenuhi persamaan abcd + 1 = (ac + 1)(bd + 1) !

Jawab :

abcd + 1 = (ac + 1)(bd + 1)

1000a + 100b + 10c + d + 1 = (10a + c + 1)(10b + d + 1)

= 100ab + 10ad + 10a + 10bc + cd + c + 10b + d + 1

990a + 90b + 9c - 100ab - 10ad - 10bc – cd = 0

(900a – 100ab) + (90a – 10ad) + (90b – 10 bc) + 9c – cd = 0

100a (9 – b) + 10a (9 – d) + 10b (9 – c) + c (9 – d) = 0

Jadi : b = d = c = 9

a = 1, 2, 3, …., 9

Sehingga bilangan-bilangan itu : 1999, 2999, 3999, …., 9999

24. Tentukan nilai dari $\frac{3}{1x2x2^1} + \frac{4}{2x3x2^2} + \frac{5}{3x4x2^3} + ... + \frac{2005}{2003x2004x2^{2003}}$

Jawab :

$\frac{k+2}{k.(k+1).2^k} = \frac{a}{2^k.k} - \frac{b}{2^k.(k+1)} = \frac{a(k+1) - kb}{k(k+1).2^k} = \frac{(a-b)k + a}{k(k+1).2^k}$

jadi a – b = 1 karena a = 2 maka b = 1

$\sum_{k=1}^{2003}\left(\frac{2}{2^k.k} - \frac{1}{2^k.(k+1)}\right) = \left(\frac{2}{2^1.1} - \frac{1}{2^1.2}\right) + \left(\frac{2}{2^2.2} - \frac{1}{2^2.3}\right) + ... + \left(\frac{2}{2^{2003}.2003} - \frac{1}{2^{2003}.2004}\right)$

$= 1 - \frac{1}{2^{2003}.2004}$

25. Jika x dan y bilangan asli yang memenuhi persamaan xy + x + y = 71 dan $x^2y + xy^2 = 880$ maka tentukan nilai $x^2 + y^2$!

Jawab :

Misal xy = a dan x + y = b maka :

xy + x + y = 71 ⇔ a + b = 71 ⇔ a = 71 – b ….. (1)

$x^2y + xy^2 = 880$ ⇔ xy(x+y) = 880 ⇔ ab = 880 …. (2)

Dari (1) dan (2) didapat :

i. b = 55 dan a = 16 maka $x^2 + y^2 = (x+y)^2 - 2xy = 55^2 - 2.16 = 2993$

ii. b = 16 dan a = 55 maka $x^2 + y^2 = (x+y)^2 - 2xy = 16^2 - 2.55 = 146$

26. Tentukan nilai A^2 dimana A adalah jumlah dari nilai mutlak semua akar-akar persamaan :

$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{9}{x}}}}$$

Jawab :

$x = \sqrt{19} + \dfrac{91}{x} \quad \Leftrightarrow \quad x^2 - \sqrt{19}x - 91 = 0$

$x_{1.2} = \dfrac{\sqrt{19} \pm \sqrt{383}}{2}$

$A = \left|\dfrac{\sqrt{19} + \sqrt{383}}{2}\right| + \left|\dfrac{\sqrt{19} - \sqrt{383}}{2}\right| = \dfrac{\sqrt{19} + \sqrt{383}}{2} + \dfrac{\sqrt{383} - \sqrt{19}}{2} = \sqrt{383}$

$A^2 = 383$

27. Didefinisikan $f(n) = \dfrac{1}{\sqrt[3]{n^2 + 2n + 1} + \sqrt[3]{n^2 - 1} + \sqrt[3]{n^2 - 2n + 1}}$ untuk semua n bilangan asli. Tentukan nilai dari f(1) + f(3) + f(5) + …. + f(999999) !

Jawab :

$x - y = \sqrt[3]{x}^3 - \sqrt[3]{y}^3 = \left(\sqrt[3]{x} - \sqrt[3]{y}\right)\left(\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}\right) \Leftrightarrow \dfrac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} = \dfrac{\sqrt[3]{x} - \sqrt[3]{y}}{x - y}$

Misal :

$x^2 = n^2 + 2n + 1 = (n+1)^2 \quad \Rightarrow \quad x = n + 1$

$y^2 = n^2 - 2n + 1 = (n-1)^2 \quad \Rightarrow \quad x = n - 1$

$xy = (n+1)(n-1) = n^2 - 1 \quad \Rightarrow \quad xy = n^2 - 1$

$f(n) = \dfrac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} = \dfrac{\sqrt[3]{x} - \sqrt[3]{y}}{x - y}$

$f(n) = \dfrac{\sqrt[3]{n+1} - \sqrt[3]{n-1}}{(n+1) - (n-1)} = \dfrac{\sqrt[3]{n+1} - \sqrt[3]{n-1}}{2}$

f(1) + f(3) + f(5) + …. + f(999999)

$= \dfrac{\left(\sqrt[3]{2} - \sqrt[3]{0}\right) + \left(\sqrt[3]{4} - \sqrt[3]{2}\right) + \left(\sqrt[3]{6} - \sqrt[3]{4}\right) + \left(\sqrt[3]{8} - \sqrt[3]{6}\right) + ... + \left(\sqrt[3]{1000000} - \sqrt[3]{999999}\right)}{2}$

$= \dfrac{0 + 100}{2} = 50$

28. Carilah 3 bilangan asli x, y, z dimana $z < y < x < 2004$ dan memenuhi persamaan $x^3 + y^4 = z^5$!

Jawab :

$x^3 = z^5 - y^4$

Misal $z = a^5$ *dan* $y = a^6$ *maka* $x^3 = a^{25} - a^{24} = a^{24}(a - 1) \quad \Leftrightarrow \quad x = a^8 \sqrt[3]{a - 1}$

a – 1 harus bilangan pangkat 3 seperti 1, 8, 27 dsb.

Misal a = 2 maka $x = 2^8 \sqrt[3]{2 - 1} = 256$

$z = 2^5 = 32$

$y = 2^6 = 64$

29. Tunjukkan bahwa untuk setiap bilangan asli n berlaku $121^n - 25^n + 1900^n - (-4)^n$ selalu habis digai 2000!

Jawab :

2000 = 125 x 16

Gunakan teori $a^n - b^n$ habis dibagi a – b

$121^n - 25^n + 1900^n - (-4)^n \quad = \quad 1900^n - 25^n + 121^n - (-4)^n$

$121^n - 25^n$ → habis dibagi 16 $\quad$ $1900^n - (-4)^n$ → habis dibagi 16 $\quad$ $1900^n - 25^n$ → habis dibagi 125 $\quad$ $121^n - (-4)^n$ → habis dibagi 125

Jadi $121^n - 25^n + 1900^n - (-4)^n$ habis dibagi 125 x 16 = 2000

30. Buktikan bahwa $1998 + 1999 \times 2^{2004}$ habis dibagi 7 !

Bukti :

$1998 + 1999 \times 2^{2004} = (7n+3) + (7n+4) \times (7+1)^{668}$

Kita lihat satuannya : $3 + 4 \times 1^{668} = 3 + 4 = 7$

Jadi $1998 + 1999 \times 2^{2004}$ habis dibagi 7

31. Tentukan 3 bilangan asli x, y, z sehingga $\frac{x^3+y^3}{x^3+z^3} = \frac{2006}{2005}$

Jawab :

$$\frac{x^3+y^3}{x^3+z^3} = \frac{(x+y)(x^2-xy+y^2)}{(x+z)(x^2-xz+z^2)}$$

Karena 2006 dan 2005 relatif prima, maka diantara faktor-faktor pembilang dan penyebut harus ada yang sama.

$x + y = x + z$ tidak mungkin, karena $y = z$.

$$x^2 - xy + y^2 = x^2 - xz + z^2$$

$$y^2 - z^2 = xy - xz$$

$$(y-z)(y+z) = x(y-z)$$

$$x = y + z$$

$$\frac{x+y}{x+z} = \frac{2006}{2005} \Rightarrow \frac{y+z+y}{y+z+z} = \frac{2006}{2005}$$

$$\left.\begin{aligned} 2y + z &= 2006 \\ 2z + y &= 2005 \end{aligned}\right\} \Rightarrow y = 669 \; dan \; z = 668 \; sehingga \; x = y + z = 1337$$

32. Tentukan rumus untuk $(1x1!) + (2x2!) + (3x3!) + \ldots + (n\,x\,n!)$!

Jawab :

$$1x1! = [(1+1)x1!] - (1x1!) = (2x1!) - (1x1!) = 2! - 1!$$

$$2x2! = [(2+1)x2!] - (1x2!) = (3x2!) - (1x2!) = 3! - 2!$$

$$3x3! = 4! - 3! \quad dst$$

$$(1x1!) + (2x2!) + (3x3!) + \ldots. + (n\,x\,n!) = (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)$$

$$= -1! + (n+1)!$$

$$= (n+1)! - 1! = (n+1)! - 1$$

33. Diketahui $\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots. - \frac{1}{1336}$ dimana a relatif prima dengan b. Tunjukkan bahwa a adalah kelipatan dari 2005!

Jawab :

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots. - \frac{1}{1336} = (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1336}) - 2(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1336})$$

$$= (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1336}) - (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{668})$$

$$= \frac{1}{669} + \frac{1}{670} + \ldots. + \frac{1}{1336}$$

$$= \left(\frac{1}{669} + \frac{1}{1336)}\right) + \left(\frac{1}{670} + \frac{1}{1335}\right) + \ldots. + \left(\frac{1}{1002} + \frac{1}{1003}\right)$$

$$= \frac{1336 + 669}{669.1336} + \frac{1335 + 670}{670.1335} + \ldots. + \frac{1003 + 1002}{1002.1003}$$

$$= \frac{2005}{669.1336} + \frac{2005}{670.1335} + \ldots. + \frac{2005}{1002.1003}$$

$$= 2005(\frac{1}{669.1336} + \frac{1}{670.1335} + \ldots. + \frac{1}{1002.1003})$$

Jadi kelipatan 2005.

34. Jika $x = 2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{x}}}}$ maka tentukan nilai x !

Jawab :

$x = 2 + \frac{3}{x} \quad \Leftrightarrow \quad x^2 - 2x - 3 = 0 \quad \Leftrightarrow \quad (x-3)(x+1) = 0 \quad \Rightarrow \quad x = 3$ yang memenuhi.

35. Diketahui $a = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + ... + \frac{1002^2}{2003}$ dan $b = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + ... + \frac{1002^2}{2005}$

Tunjukkan bilangan bulat terdekat dari a – b !

Jawab :

$$a - b = (\frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + ... + \frac{1002^2}{2003}) - (\frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + ... + \frac{1002^2}{2005})$$

$$= \frac{1^2}{1} + \left(\frac{2^2}{3} - \frac{1^2}{3}\right) + \left(\frac{3^2}{5} - \frac{2^2}{5}\right) + ... + \left(\frac{1002^2}{2003} - \frac{1001^2}{2003}\right) - \frac{1002^2}{2005}$$

$$= 1 - \frac{1002^2}{2005} + (1 + 1 + 1 + ... + 1)$$

$$= 1002 - \frac{1002^2}{2005} = \frac{1002(2005 - 1002)}{2005} = \frac{1002.1003}{2005} \approx 501$$

36. Diketahui a, b, c, d, e dan f adalah bilangan real. Jika $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = 64$ maka tentukan

$$\sqrt{\frac{5a^2c - 4c^2e + e^3}{5b^2d - 4d^2f + f^3}}$$

Jawab :

$\frac{a}{b} = 64 \Leftrightarrow \quad a = 64b$

$\frac{c}{d} = 64 \Leftrightarrow \quad c = 64d$

$\frac{e}{f} = 64 \Leftrightarrow \quad e = 64f$

$$\sqrt{\frac{5a^2c - 4c^2e + e^3}{5b^2d - 4d^2f + f^3}} = \sqrt{\frac{5(64b)^2.64d - 4(64d)^2.64f + (64f)^3}{5b^2d - 4d^2f + f^3}}$$

$$= \sqrt{\frac{64^3(5b^2d - 4d^2f + f^3)}{5b^2d - 4d^2f + f^3}} = \sqrt{64^3} = 512$$

37. Diketahui $A = \sum_{k=1}^{2004}\left(\frac{1}{1 + 2 + 3 + ... + k}\right)$. Tentukan nilai A !

Jawab :

$$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$$

$$\sum_{k=1}^{2004} \frac{1}{1 + 2 + 3 + ... + k} = \sum_{k=1}^{2004} \frac{2}{k(k+1)} = \sum_{k=1}^{2004} \frac{2}{k} - \frac{2}{k+1}$$

$$= \left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + ... + \left(\frac{2}{2004} - \frac{2}{2005}\right) = 2 - \frac{2}{2005} = \frac{4008}{2005}$$

38. Jika $f(x)+2f\left(\frac{1}{x}\right)=3x \quad dan \quad x \neq 0$ maka tentukan penyelesaian untuk f(x) = f(-x) !

Jawab :

$f(x)+2f(\frac{1}{x})=3x$ ......(1)

$\Rightarrow \; f(\frac{1}{x})+2f(x)=\frac{3}{x}$ ....(2)

$Jika\ f(\frac{1}{x})\ dihilangkan\ maka\ f(x)=\frac{2-x^2}{x}$

$f(x)=f(-x)\Rightarrow \frac{2-x^2}{x}=\frac{2-x^2}{-x}\Rightarrow x=\pm\sqrt{2}$

39. Tentukan nilai dari x^3+y^3 jika diketahui $x+y+\frac{x}{y}=19\ dan\ \frac{x^2+xy}{y}=60$!

Jawab :

Misal x + y = a dan $\frac{x}{y}=b$ maka :

$x+y+\frac{x}{y}=19\Rightarrow$ a + b = 19 atau a = 19 – b ......(1)

$\frac{x^2+xy}{y}=60\Leftrightarrow \frac{x}{y}(x+y)=60\Rightarrow ab=60$.........(2)

Dari (1) dan (2) didapat :

i. b = 4 dan a = 15 maka x + y = 15 dan x = 4y sehingga x = 12 dan y = 3 jadi $x^3+y^3=1755$

ii. b=15 dan a = 4 maka x + y = 4 dan x = 15y sehingga x = 15/4 dan y = ¼ jadi $x^3+y^3=\frac{3376}{64}$

40. Tentukan penyelesaian (x,y,z) dari sistem persamaan : $\begin{cases} x+y+xy=11 \\ y+z+yz=14 \\ z+x+zx=19 \end{cases}$

Jawab :

$x+y+xy=11\Leftrightarrow y=\frac{11-x}{x+1}$

$z+x+zx=19\Leftrightarrow z=\frac{19-x}{x+1}$

$y+z+yz=14\Rightarrow \frac{11-x}{x+1}+\frac{19-x}{x+1}+\left(\frac{11-x}{x+1}\right)\left(\frac{19-x}{x+1}\right)=14\Leftrightarrow x^2+2x-15=0$

$x=3\Rightarrow y=2, z=4$

$x=-5\Rightarrow y=-4, z=-6$

41. Jika x, y, z adalah bilangan real yang memenuhi persamaan :

$x+y+z=1$

$x^2+y^2+z^2=2$

$x^3+y^3+z^3=3$

Maka tentukan nilai $x^4+y^4+z^4$!

Jawab :

$(x+y+z)^2=x^2+y^2+z^2+2(xy+xz+yz) \quad \Leftrightarrow \quad 1^2=2+2(xy+xz+yz)$

$xy+xz+yz=-\frac{1}{2}$

$$\left(x+y+z\right)^3 = x^3+y^3+z^3+3(xy+xz+yz)(x+y+z)-3xyz$$

$$1^3 = 3+3(-\frac{1}{2}).1-3xyz \quad \Leftrightarrow \quad xyz = \frac{1}{6}$$

$$\left(x^2+y^2+z^2\right)^2 = x^4+y^4+z^4+2(x^2y^2+x^2z^2+y^2z^2)$$

$$= x^4+y^4+z^4+2\left[(xy)^2+(xz)^2+(yz)^2\right]$$

$$= x^4+y^4+z^4+2\left[(xy+xz+yz)-2xyz(x+y+z)\right]$$

$$2^2 = x^4+y^4+z^4+2\left[\left(-\frac{1}{2}\right)^2-2.\frac{1}{6}.1\right]$$

$$x^4+y^4+z^4 = 4\frac{1}{6}$$

42. Diketahui $f(x) = (x+3)^4 - 12(x+3)^3 + 54(x+3)^2 - 108(x+3) + 81$. Tulislah f(x) dalam bentuk yang paling sederhana dan tentukan f(2005)!

Jawab :

$$x^4 = [(x+3)-3]^4 = (x+3)^4 - 4(x+3)^3.3 + 6(x+3)^2.3^2 - 4(x+3).3^3 + 3^4$$

$$= (x+3)^4 - 12(x+3)^3 + 54(x+3)^2 - 108(x+3) + 81$$

$$= x^4$$

$$f(2005) = 2005^4$$

43. Tentukan nilai x, y, z yang memenuhi persamaan $\frac{xy}{x+y} = \frac{1}{2}, \quad \frac{yz}{y+z} = \frac{1}{3}, \quad \frac{zx}{z+x} = \frac{1}{7}$

Jawab :

$$\frac{xy}{x+y} = \frac{1}{2} \Leftrightarrow \frac{x+y}{xy} = 2 \Leftrightarrow \frac{1}{x}+\frac{1}{y} = 2 \Rightarrow a+b = 2 \quad ....(1)$$

$$\frac{yz}{y+z} = \frac{1}{3} \Leftrightarrow \frac{y+z}{yz} = 3 \Leftrightarrow \frac{1}{y}+\frac{1}{z} = 3 \Rightarrow b+c = 3 \quad ....(2)$$

$$\frac{zx}{z+x} = \frac{1}{7} \Leftrightarrow \frac{z+x}{zx} = 7 \Leftrightarrow \frac{1}{x}+\frac{1}{z} = 7 \Rightarrow a+c = 7 \quad ....(3)$$

Dari (1), (2) *dan* (3) *didapat* :

$$a = 3 = \frac{1}{x} \Leftrightarrow x = \frac{1}{3}$$

$$b = -1 = \frac{1}{y} \Leftrightarrow y = -1$$

$$c = 4 = \frac{1}{z} \Leftrightarrow z = \frac{1}{4}$$

44. Tentukanlah nilai dari $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)....\left(1-\frac{1}{2004}\right)$!

Jawab :

$$\frac{1}{2}.\frac{2}{3}.\frac{3}{4}.......\frac{2002}{2003}.\frac{2003}{2004} = \frac{1}{2004}$$

45. Tentukan nilai dari $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{2004.2005}$!

Jawab :

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots\dots+\frac{1}{2004.2005}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots\dots+\left(\frac{1}{2004}-\frac{1}{2005}\right)$$
$$=1-\frac{1}{2005}=\frac{2004}{2005}$$

46. Tentukan nilai dari $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\dots+\frac{1}{\sqrt{9999}+\sqrt{10000}}$!

Jawab:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\dots+\frac{1}{\sqrt{9999}+\sqrt{10000}}$$
$$=\left(-\sqrt{1}+\sqrt{2}\right)+\left(-\sqrt{2}+\sqrt{3}\right)+\left(-\sqrt{3}+\sqrt{4}\right)+\dots\dots+\left(-\sqrt{9999}+\sqrt{10000}\right)=-\sqrt{1}+\sqrt{10000}=99$$

47. Jika $\frac{57}{17}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+1}}}$ maka tentukan nilai a x b x c x d !

Jawab:

$$\frac{57}{17}=3+\frac{6}{17}=3+\frac{1}{\frac{17}{6}}=3+\frac{1}{2+\frac{5}{6}}=3+\frac{1}{2+\frac{1}{\frac{6}{5}}}=3+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{5}}}=3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+1}}}$$

Jadi a = 3, b = 2, c = 1 dan d = 4

Sehingga a x b x c x d = 3.2.1.4 = 24

47. Jika $\frac{x+2y}{6}=\frac{2y+3z}{10}=\frac{3z+x}{8}$ maka tentukan nilai dari $\frac{2005yz+2005zx+2005xy}{x^2+y^2+z^2}$!

Jawab:

$10x+20y=12y+18z \Leftrightarrow 5x+4y-9z=0$ ....(1)

$8x+16y=18z+6x \Leftrightarrow x+8y-9z=0$ ....(2)

$16y+24z=30z+10x \Leftrightarrow 5x-8y+3z=0$ ....(3)

dari (1), (2) dan (3) didapat x = y = z

$$\frac{2005yz+2005zx+2005xy}{x^2+y^2+z^2}=\frac{2005\left(x^2+x^2+x^2\right)}{x^2+x^2+x^2}=2005$$

48. Diketahui:

$$A=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\dots+\frac{1}{2003}-\frac{1}{2004}$$
$$B=\frac{1}{1003}+\frac{1}{1004}+\frac{1}{1005}+\dots\dots+\frac{1}{2004}$$

Maka hitunglah nilai dari A^2-B^2 !

Jawab:

$$A=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\dots+\frac{1}{2003}-\frac{1}{2004}$$
$$=\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2004}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots\dots+\frac{1}{2004}\right)$$
$$=\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2004}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{1002}\right)$$
$$=\frac{1}{1003}+\frac{1}{1004}+\dots\dots+\frac{1}{2004}$$

Jadi A = B maka $A^2-B^2=0$

50. Buktikan bahwa $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{2005!}<2$!

Jawab :

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{2005!}<\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+....+\frac{1}{2^{2004}}=\frac{\frac{1}{2}(1-\left(\frac{1}{2}\right)^{2004})}{1-\frac{1}{2}}=1-\left(\frac{1}{2}\right)^{2004}$$

$$Karena\left(\frac{1}{2}\right)^{2004}>0\ maka\ 1-\left(\frac{1}{2}\right)^{2004}<1$$

$$Jadi\ \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{2005!}<1+1=2$$

51. Tentukan nilai dari $\frac{1}{e^{-2005}+1}+\frac{1}{e^{-2004}+1}+....+\frac{1}{2}+....+\frac{1}{e^{2004}+1}+\frac{1}{e^{2005}+1}$

Jawab :

$$\frac{1}{e^{-x}+1}+\frac{1}{e^{x}+1}=\frac{e^x+e^{-x}+2}{e^x+e^{-x}+2}=1$$

$$\frac{1}{e^{-2005}+1}+\frac{1}{e^{-2004}+1}+....+\frac{1}{2}+....+\frac{1}{e^{2004}+1}+\frac{1}{e^{2005}+1}$$

$$=\left(\frac{1}{e^{-2005}+1}+\frac{1}{e^{2005}+1}\right)+\left(\frac{1}{e^{-2004}+1}+\frac{1}{e^{2004}+1}\right)+......+\frac{1}{e^0+1}$$

$$=1+1+1+......+1+\frac{1}{1+1}$$

$$=2005\frac{1}{2}$$

52. Diketahui a, b, c, d adalah bilangan-bilangan real yang memenuhi persamaan :

$a+4b+9c+16d=52$ …………..(1)

$4a+9b+16c+25d=150$ ………...(2)

$9a+16b+25c+36d=800$ ………(3)

Tentukan nilai dari $16a+25b+36c+49d$!

Jawab :

$$(n+3)^2-3(n+2)^2+3(n+1)^2-n^2=0$$

$$(n+3)^2=3(n+2)^2-3(n+1)^2+n^2$$

$$(1+3)^2a=3(1+2)^2a-3(1+1)^2a+1^2.a\Leftrightarrow 27a-12a+a=16a$$

$$(2+3)^2b=3(2+2)^2b-3(2+1)^2b+2^2.b\Leftrightarrow 48b-27b+4b=25b$$

$$Pers.(3)x3\Rightarrow 27a+48b+75c+108d=2400$$

$$Pers.(2)x3\Rightarrow 12a+27b+48c+75d=450$$

$$\overline{\quad 15a+21b+27c+33d=1950\quad}\ -$$

Persa.(1)x1⇒ $a+4b+9c+16d=52$

$$\overline{\quad 16a+25b+36c+49d=2002\quad}\ +$$

53. Jika $x=\frac{1+\sqrt{2004}}{2}$ maka tentukan nilai dari:

$a)\ 4x^3-2007x-2000$

$b)\ 4x^{2005}-4x^{2004}-2003x^{2003}$

Jawab :

$a)\ x = \dfrac{1+\sqrt{2004}}{2} \Leftrightarrow 4x^2 = 4x + 2003 \Rightarrow 4x^3 = 4x^2 + 2003x$

$4x^3 - 2007x - 2000 = 4x^2 + 2003x - 2007x - 2000 = 4x^2 - 4x - 2000$

$= 4x + 2003 - 4x - 2000 = 3$

$b)\ 4x^2 - 4x + 1 = 2004 \Leftrightarrow 4x^2 - 4x - 2003 = 0 \quad \left|.x^{2003}\right|$

$4x^{2005} - 4x^{2004} - 2003x^{2003} = 0$

54. Jika a dan b adalah bilangan real yang memenuhi persamaan :

$\dfrac{1}{ab} + a + b = 11$

$a^2b^2(a+b)^2 = 61a^2b^2 - 1$

Tentukan nilai dari $\dfrac{1}{a} + \dfrac{1}{b}$!

Jawab :

Misal $\dfrac{1}{ab} = x \quad dan \quad a + b = y$

$\dfrac{1}{ab} + a + b = 11 \Rightarrow x + y = 11 \quad ......(1)$

$a^2b^2(a+b)^2 = 61a^2b^2 - 1 \Rightarrow \dfrac{1}{x^2}.y^2 = \dfrac{61}{x^2} - \dfrac{x^2}{x^2} \Leftrightarrow y^2 = 61 - x^2 \quad .....(2)$

Dari (1) *dan* (2) *didapat* :

$i)\quad x = 6 = \dfrac{1}{ab} \Leftrightarrow ab = \dfrac{1}{6}$

$y = 5 = a + b$

$\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab} = \dfrac{5}{\frac{1}{6}} = 30$

$ii)\quad x = 5 = \dfrac{1}{ab} \Leftrightarrow ab = \dfrac{1}{5}$

$y = 6 = a + b$

$\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab} = \dfrac{6}{\frac{1}{5}} = 30$

55. Buktikan bahwa jika suatu bilangan kelipatan 9 maka jumlah angka-angkanya pasti kelipatan 9 !

Jawab :

abcd = 1000a + 100b + 10c + d = 999a + a + 9b + b + 9c + c + d

= 999a + 99b + 9c + (a+b + c + d)

= 9(111a + 11b + c) + (a + b + c + d)

a + b + c + d = abcd – 9(111a + 11b + c)

Karena abcd kelipatan 9 maka a + b + c + d kelipatan 9.

56. Diketahui bilangan asli berurutan a, b, c , d. buktikan bahwa ab + ac + ad + bc + bd + cd + 1 habis dibagi 12

Jawab :

Hasil kali 2 bilangan asli berurutan pasti bilangan genap (kelipatan 2)

Misal a = x, b = x + 1, c = x + 2 dan d = x + 3

ab + ac + ad + bc + bd + cd + 1

=x(x+1)+x(x+2)+x(x+3)+(x+1)(x+2)+(x+1)(x+3)+(x+2)(x+3)+1

$=x^2 +x+x^2 +2x+x^2 +3x+x^2 +3x+2+x^2 +4x+3+x^2 +5x+6+1$

$=6x^2 +6x+12x+12$

=6x(x+1) + 12(x+1)
Karena x(x+1) kelipatan 2 maka 6x(x+1) kelipatan 12
Jadi soal kelipatan 12.

57. Buktikan bahwa semua bilangan asli yang terdiri dari 6 digit angka berbentuk abcabc selalu habis dibagi 91 !

Bukti :
abcabc = 100.000a + 10.000b + 1.000c + 100a + 10b + c
= 100100a + 10010b + 1001c
= 1001 x 100a + 1001 x 10b + 1001c
= (91 x 11) x 100a + (91 x 11) x 10b + (91 x 11)c
Jadi abcabc habis dibagi 91.

58. Jika a, b, c bilangan real positif dan a + b + c = 1, buktikan bahwa $(1 - a)(1 - b)(1 - c) \geq 8abc$!

Jawab :

$a + b + c = 1$

$a + b = 1 - c$

$b + c = 1 - a$

$a + c = 1 - b$

$a + b \geq 2\sqrt{ab}$ ....(1)

$b + c \geq 2\sqrt{bc}$ ....(2)

$a + c \geq 2\sqrt{ac}$ ....(3)

Jika $(1)x(2)x(3)$ *maka* :

$(a + b)(b + c)(a + c) \geq 8\sqrt{a^2b^2c^2}$

$(1 - a)(1 - b)(1 - c) \geq 8abc$

59. Jika A = 1 + (1+2) + (1+2+4) + (1+2+4+8) + …..+(1+2+4+…..+2^{n-1}) maka tentukan rumus untuk nilai A !

Jawab :

1 + (1+2) + (1+2+4) + (1+2+4+8) + …..+(1+2+4+…..+2^{n-1}) $= \sum_{i=1}^{n}\left(\sum_{k=1}^{i} 2^{k-1}\right)$

Karena 1+2+4+…… ⇒ $S_i = \dfrac{1(2^i - 1)}{2 - 1} = 2^i - 1$ maka :

1 + (1+2) + (1+2+4) + (1+2+4+8) + …..+(1+2+4+…..+2^{n-1})

$= \sum_{i=1}^{n} 2^i - 1 = \sum_{i=1}^{n} 2^i - \sum_{i=1}^{n} 1 = \sum_{i=1}^{n} 2^i - n$

$\sum_{i=1}^{n} 2^i = 2 + 4 + 8 + ... = \dfrac{2(2^n - 1)}{2 - 1} = 2^{n+1} - 2$

$\sum_{i=1}^{n} 2^i - 1 = 2^{n+1} - 2 - n = 2^{n+1} - (n + 2)$

60. Jika a, b dan c bilangan real positif, buktikan bahwa $a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2 \geq 6abc$!

Jawab :

$a^2 + c^2 \geq 2ac \quad |.b| \Rightarrow a^2b + bc^2 \geq 2abc$ ....(1)

$b^2 + c^2 \geq 2bc \quad |.a| \Rightarrow ab^2 + ac^2 \geq 2abc$ ....(2)

$a^2 + b^2 \geq 2ab \quad |.c| \Rightarrow a^2c + b^2c \geq 2abc$ ....(3)

Jika $(1) + (2) + (3)$ *maka* $a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2 \geq 6abc$

61. Tentukan penyelesaian dari sistem persamaan berikut jika diketahui a, b, c, d bilangan real.

$abc + ab + bc + ca + a + b + c = 1$ …. (1)
$bcd + bc + cd + db + b + c + d = 9$ …. (2)
$cda + cd + da + ac + c + d + a = 9$ …. (3)
$dab + da + ab + bd + d + a + b = 9$ …. (4)

Jawab :

Jika pers (1), (2), (3) dan (4) semua masing-masing ruas di tambah 1, maka didapat :

$(a+1)(b+1)(c+1) = 2$
$(b+1)(c+1)(d+1) = 10$
$(a+1)(c+1)(d+1) = 10$
$(a+1)(b+1)(d+1) = 10$
——————————— x

$[(a+1)(b+1)(c+1)(d+1)]^3 = 2000$

$(a+1)(b+1)(c+1)(d+1) = 10\sqrt[3]{2}$

$2(d+1) = 10\sqrt[3]{2} \Leftrightarrow d = 5\sqrt[3]{2} - 1$

$10(a+1) = 10\sqrt[3]{2} \Leftrightarrow a = \sqrt[3]{2} - 1$

$10(b+1) = 10\sqrt[3]{2} \Leftrightarrow b = \sqrt[3]{2} - 1$

$10(c+1) = 10\sqrt[3]{2} \Leftrightarrow c = \sqrt[3]{2} - 1$

62. Jika a dan b bilangan real dan $\frac{a}{b} + \frac{a+10b}{b+10a} = 2$ maka tentukan nilai $\frac{a}{b}$!

Jawab :

$\frac{a}{b} + \frac{a+10b}{b+10a} = 2 \Leftrightarrow \frac{a}{b} + \frac{\frac{a}{b}+10}{1+10\frac{a}{b}} = 2 \Leftrightarrow 5\left(\frac{a}{b}\right)^2 - 9\left(\frac{a}{b}\right) + 4 = 0$

$\Leftrightarrow \left(5\frac{a}{b} - 4\right)\left(\frac{a}{b} - 1\right) = 0 \Rightarrow \frac{a}{b} = \frac{4}{5} \; atau \; \frac{a}{b} = 1$

63. Jika a, b, c dan d real positif dan berlaku $\frac{a}{b} < \frac{c}{d}$ maka buktikan bahwa $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

Jawab :

$ab + ad < ab + bc$ $\qquad$ $ad + cd < bc + cd$

$a(b+d) < b(a+c)$ $\qquad$ $d(a+c) < c(b+d)$

$\frac{a}{b} < \frac{a+c}{b+d}$ ......(1) $\qquad$ $\frac{a+c}{b+d} < \frac{c}{d}$ ....(2)

Dari (1) *dan* (2) : $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

64. Buktikan bahwa untuk n bilangan bulat maka $n^3 + 2n$ selalu habis dibagi 3 !

Jawab :

$n^3 + 2n = n(n^2+2) = n(n^2 - 1 + 3) = n(n^2-1) + 3n = (n-1)n(n+1) + 3n$

Karena (n-1)n(n+1) merupakan 3 bilangan berurutan maka (n-1)n(n+1) habis dibagi 3, jadi $n^3 + 2n$ habis dibagi 3.

65. Tentukan nilai x, y, z bilangan real yang memenuhi persamaan :

$x^2 + 2yz = x$ ....(1)

$y^2 + 2zx = y$ ....(2)

$z^2 + 2xy = z$ ....(3)

Jawab :
Jika pers (1) kali x, pers (2) kali y dan pers (3) kali z maka didapat :

$x^3 + 2xyz = x^2$

$y^3 + 2xyz = y^2$

$z^3 + 2xyz = z^2$

Dengan mengeliminasi 2xyz maka didapat x = y = z

$x^2 + 2yz = x \Rightarrow x^2 + 2x.x = x \Rightarrow x = \frac{1}{3} = y = z$

66. Jika a, b, c real positif sedemikian sehingga $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ tentukan nilai dari $\frac{2003a + 2004b + 2005c}{3a + 2b + c}$

Jawab :

$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ maka a = b = c

$\frac{2003a + 2004b + 2005c}{3a + 2b + c} \Rightarrow \frac{2003a + 2004a + 2005a}{3a + 2a + a} = 1002$

67. Buktikan bahwa untuk n bilangan asli yang lebih dari 1 maka $n^5 - n$ habis dibagi 30 !

Jawab :

$n^5 - n = (n-1)n(n+1)(n^2 + 1$

(n-1)n(n+1) habis dibagi 6.

Bilangan yang habis dibagi 5 sellau berujung 5 atau 0.

$n^5 - n = n(n^2 - n)(n^2 + 1)$

Untuk n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 selalu berujung 0 atau 5, jadi habis dibagi 5.

Sehingga $n^5 - n$ habis dibagi 6 x 5 = 30

68. Buktikan bahwa $11^{10} - 1$ habis dibagi 100 !

Bukti :

$11^{10} - 1 = (10+1)^{10} - 1 = 10^{10} + 10.10^9 + 45.10^8 + 120.10^7 + 210.10^6 + 252.10^5 + 210.10^4 + 120.10^3 + 45.10^2 + 10.10^1 + 1 - 1$

Habis dibagi 100.

69. Tentukan nilai positif x, y, z dari persamaan :

${}^2\log x + {}^4\log y + {}^4\log z = 2$ ....(1)

${}^3\log y + {}^9\log z + {}^9\log x = 2$ ....(2)

${}^4\log z + {}^{16}\log x + {}^{16}\log y = 2$ ....(3)

Jawab :

${}^2\log x + {}^4\log y + {}^4\log z = 2 \Leftrightarrow {}^4\log x^2 + {}^4\log y + {}^4\log z = {}^4\log 16 \Rightarrow x^2yz = 16$ ...(1)

${}^3\log y + {}^9\log z + {}^9\log x = 2 \Leftrightarrow {}^9\log y^2 + {}^9\log z + {}^9\log x = {}^9\log 81 \Rightarrow y^2zx = 81$ ...(2)

${}^4\log z + {}^{16}\log x + {}^{16}\log y = 2 \Leftrightarrow {}^{16}\log z^2 + {}^{16}\log x + {}^{16}\log y = {}^{16}\log 256 \Rightarrow z^2xy = 256$ ...(3)

$(1)x(2)x(3) \Rightarrow (xyz)^4 = 16.81.256 \Rightarrow xyz = 24$

$x^2yz = 16 \Leftrightarrow x.xyz = 16 \Rightarrow x.24 = 16 \Leftrightarrow x = \frac{2}{3}$

$y^2zx = 81 \Leftrightarrow y.xyz = 81 \Rightarrow y.24 = 81 \Leftrightarrow y = \frac{27}{8}$

$z^2xy = 256 \Leftrightarrow z.xyz = 256 \Rightarrow z.24 = 256 \Leftrightarrow z = \frac{32}{3}$

70. Diketahui a+b+c+d=0 dan a.b.c.d≠ 0. Buktikan bahwa:

$\left(a^3+b^3+c^3+d^3\right)^2=9\left(abc+abd+acd+bcd\right)^2$

Bukti :

$a+b+c+d=0 \Leftrightarrow a+b=-(c+d)$

$a^3+b^3=(a+b)^3-3ab(a+b)$

$=-(c+d)^3+3ab(c+d)$

$a^3+b^3=-c^3-d^3-3cd(c+d)+3abc+3abd$

$a^3+b^3+c^3+d^3=3cd(a+b)+3abc+3abd=3(acd+bcd+abc+abd)$

$(a^3+b^3+c^3+d^3)^2=9(acd+bcd+abc+abd)^2$

71. Diketahui x dan y adalah bilangan real dengan ketentuan $1<y<2 \; dan \; x-y+1=0$. Tentukan nilai dari $\sqrt{4x^2+4y-3}+2\sqrt{y^2-6x-2y+10}$

Jawab :

x = y – 1 disubstitusikan ke $\sqrt{4x^2+4y-3}+2\sqrt{y^2-6x-2y+10}$ maka akan didapat:

$\sqrt{(2y-1)^2}+2\sqrt{(y-4)^2}=|2y-1|+2|y-4|$

Karena $1<y<2$ maka:

$|2y-1|=2y-1$

$|y-4|=-(y-4)$

$Jadi \quad 2y-1-2(y-4)=7$

72. Sebuah bilangan terdiri dari 3 digit. Bilangan itu habis dibagi 12 dan hasil baginya adalah jumlah angka-angkanya. Tentukan bilangan itu!

Jawab :

100a+10b+c=12(a+b+c) $\Leftrightarrow 44a=5\frac{1}{2}c+b$

c yang mungkin adalah bilangan genap yaitu 8

44a = $5\frac{1}{2}$.8 + b atau b = 44a – 44 maka a = 1 dan b = 0

Jadi bilangan itu adalah 108

73. Dalam segitiga ABC, buktikan bahwa $\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C\le\frac{1}{8}$!

Bukti :

$$\sin^2\frac{1}{2}A=\frac{1-\cos A}{2}=\frac{1-\dfrac{b^2+c^2-a^2}{2bc}}{2}=\frac{a^2-(b-c)^2}{4bc}$$

$Karena \; (b-c)^2\ge 0 \; maka \; a^2-(b-c)^2\le a^2.$

$$Sehingga \; \sin^2\frac{1}{2}A\le\frac{a^2}{4bc} \; akibatnya \; \sin^2\frac{1}{2}B\le\frac{b^2}{4ac} \; dan \; \sin^2\frac{1}{2}C\le\frac{c^2}{4ab}$$

$$\sin^2\frac{1}{2}A.\sin^2\frac{1}{2}B.\sin^2\frac{1}{2}C\le\frac{a^2}{4bc}.\frac{b^2}{4ac}.\frac{c^2}{4ab}$$

$$\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C\le\sqrt{\frac{a^2b^2c^2}{64a^2b^2c^2}}=\frac{1}{8}$$

74. Dalam segitiga ABC, buktikan bahwa $\cos A\cos B\cos C \le \frac{1}{8}$!

<u>Bukti</u> :

Karena $\left(b^2-c^2\right)^2 \ge 0$ maka $a^4-(b^2-c^2)^2 \le a^4$

$\left[a^2+(b^2-c^2)\right]\left[a^2-(b^2-c^2)\right] \le a^4$

$\left(2ab\cos C\right)(2ac\cos B) \le a^4$

$$\cos B\cos C \le \frac{a^2}{4bc} \textit{ akibatnya } \cos A\cos B \le \frac{c^2}{4ab} \textit{ dan } \cos A\cos C \le \frac{b^2}{4ac}$$

$$\cos A\cos B.\cos A\cos C.\cos B\cos C \le \frac{c^2}{4ab}.\frac{b^2}{4ac}.\frac{a^2}{4bc}$$

$$\cos A\cos B\cos C \le \sqrt{\frac{a^2b^2c^2}{64a^2b^2c^2}} = \frac{1}{8}$$

75. Jika A, B, C sudut-sudut pada segitiga ABC, buktikan bahwa $\sin A+\sin B+\sin C \le \frac{3}{2}\sqrt{3}$!

<u>Bukti</u> :

$$\sin A+\sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\textit{Karena } \cos\frac{1}{2}(A-B) \le 1 \textit{ maka } \sin A+\sin B \le 2\sin\frac{1}{2}(A+B) \quad ...(1)$$

$$\textit{Sehingga } \sin C+\sin 60^\circ \le 2\sin\frac{1}{2}(C+60^\circ) \quad ......(2)$$

$$\textit{Jika pers.}(1)+(2) \textit{ maka}:$$

$$\sin A+\sin B+\sin C+\frac{1}{2}\sqrt{3} \le 2\left(\sin(\frac{1}{2}A+\frac{1}{2}B)+\sin\left(\frac{1}{2}C+30^\circ\right)\right)$$

$$\sin A+\sin B+\sin C+\frac{1}{2}\sqrt{3} \le 2(2\sin\frac{1}{4}(A+B+C+60^\circ)\cos\frac{1}{4}(A+B-C-60^\circ))$$

$$\textit{Karena } \cos\frac{1}{4}(A+B-C-60^\circ) \le 1 \textit{ maka}:$$

$$\sin A+\sin B+\sin C+\frac{1}{2}\sqrt{3} \le 4\sin\frac{1}{4}(180^\circ+60^\circ)$$

$$\sin A+\sin B+\sin C \le 4.\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$$

76. Pada segitiga ABC, buktikan bahwa $\tan\frac{1}{2}A\tan\frac{1}{2}B+\tan\frac{1}{2}A\tan\frac{1}{2}C+\tan\frac{1}{2}B\tan\frac{1}{2}C = 1$

<u>Bukti</u> :

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B}+\frac{\sin\frac{1}{2}A\sin\frac{1}{2}C}{\cos\frac{1}{2}A\cos\frac{1}{2}C}+\frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C+\sin\frac{1}{2}A\sin\frac{1}{2}C\cos\frac{1}{2}B+\sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}A\sin(\frac{1}{2}B+\frac{1}{2}C)+\sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}A\sin\frac{1}{2}\left(180^\circ - A\right) + \sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}A\cos\frac{1}{2}A + \sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}A + \sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}\left(180^\circ - (B + C)\right) + \sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\cos\left(\frac{1}{2}B + \frac{1}{2}C\right) + \sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}B\cos\frac{1}{2}C - \sin\frac{1}{2}B\sin\frac{1}{2}C + \sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}B\cos\frac{1}{2}C} = 1$$

77. Jika A, B, C adalah sudut-sudut dalam segitiga ABC, buktikan bahwa :

$$\sqrt{\tan\frac{1}{2}A\tan\frac{1}{2}B + 8} + \sqrt{\tan\frac{1}{2}B\tan\frac{1}{2}C + 8} + \sqrt{\tan\frac{1}{2}A\tan\frac{1}{2}C + 8} \le 5\sqrt{3}$$

<u>Bukti</u> :

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 = a + b + c + 2\sqrt{ab} + 2\sqrt{ac} + 2\sqrt{bc}$$

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \le a + b + c + a + b + a + c + b + c = 3(a + b + c)$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \le \sqrt{3(a + b + c)}$$

$$\sqrt{\tan\frac{1}{2}A\tan\frac{1}{2}B + 8} + \sqrt{\tan\frac{1}{2}B\tan\frac{1}{2}C + 8} + \sqrt{\tan\frac{1}{2}A\tan\frac{1}{2}C + 8}$$

$$\le \sqrt{3\left(\tan\frac{1}{2}A\tan\frac{1}{2}B + \tan\frac{1}{2}A\tan\frac{1}{2}C + \tan\frac{1}{2}B\tan\frac{1}{2}C + 24\right)}$$

$$= \sqrt{3(1 + 24)} = 5\sqrt{3}$$

78. Buktikan bahwa $\cos\frac{\pi}{2005} + \cos\frac{3\pi}{2005} + \cos\frac{5\pi}{2005} + ... + \cos\frac{2003\pi}{2005} = \frac{1}{2}$

<u>Bukti</u> :

$$2\cos\frac{\pi}{2005}\sin\frac{\pi}{2005} = \sin\frac{2\pi}{2005} - 0$$

$$2\cos\frac{3\pi}{2005}\sin\frac{\pi}{2005} = \sin\frac{4\pi}{2005} - \sin\frac{2\pi}{2005}$$

$$2\cos\frac{5\pi}{2005}\sin\frac{\pi}{2005} = \sin\frac{6\pi}{2005} - \sin\frac{4\pi}{2005}$$

....

$$2\cos\frac{2003\pi}{2005}\sin\frac{\pi}{2005} = \sin\frac{2004\pi}{2005} - \sin\frac{2002\pi}{2005}$$

_____________________________________ +

$$2\sin\frac{\pi}{2005}\left(\cos\frac{\pi}{2005}+\cos\frac{3\pi}{2005}+\cos\frac{5\pi}{2005}+......+\cos\frac{2003\pi}{2005}\right)=\sin\frac{2004\pi}{2005}$$

$$\cos\frac{\pi}{2005}+\cos\frac{3\pi}{2005}+\cos\frac{5\pi}{2005}+......+\cos\frac{2003\pi}{2005}=\frac{1}{2}\frac{\sin\frac{2004\pi}{2005}}{\sin\frac{\pi}{2005}}=\frac{1}{2}\frac{\sin\left(\pi-\frac{\pi}{2005}\right)}{\sin\frac{\pi}{2005}}=\frac{1}{2}$$

79. Buktikan bahwa $\cos ec\ 10^\circ+\cos ec\ 50^\circ-\cos ec\ 70^\circ=6$

<u>Bukti</u> :

$$\cos ec\ 10^\circ+\cos ec\ 50^\circ-\cos ec\ 70^\circ=\frac{1}{\sin 10^\circ}+\frac{1}{\sin 50^\circ}-\frac{1}{\sin 70^\circ}$$

$$=\frac{\sin 70^\circ\sin 50^\circ+\sin 70^\circ\sin 10^\circ-\sin 50^\circ\sin 10^\circ}{\sin 70^\circ\sin 50^\circ\sin 10^\circ}$$

$$=\frac{-\frac{1}{2}(\cos 120^\circ-\cos 20^\circ+\cos 80^\circ-\cos 60^\circ-\cos 60^\circ+\cos 40^\circ}{-\frac{1}{2}(\cos 120^\circ-\cos 20^\circ)\sin 10^\circ}$$

$$=\frac{-\frac{3}{2}+\cos 80^\circ+\cos 40^\circ-\cos 20^\circ}{-\frac{1}{2}\sin 10^\circ-\frac{1}{2}(\sin 30^\circ-\sin 10^\circ)}$$

$$=\frac{-\frac{3}{2}+2\cos 60^\circ\cos 20^\circ-\cos 20^\circ}{-\frac{1}{2}\sin 10^\circ-\frac{1}{4}+\frac{1}{2}\sin 10^\circ}$$

$$=\frac{-\frac{3}{2}}{-\frac{1}{4}}=6$$

80. Pada segitiga ABC, buktikan bahwa $\tan A+\tan B+\tan C=\tan A\tan B\tan C$!

<u>Bukti</u> :

$$\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C}=\frac{\sin A\cos B\cos C+\sin B\cos A\cos C+\sin C\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\cos C(\sin A\cos B+\cos A\sin B)+\sin C\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\cos C\sin(A+B)+\sin C\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\cos c\sin C+\sin C\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\sin C(\cos C+\cos A\cos B)}{\cos A\cos B\cos C}$$

$$=\frac{\sin C(\cos A\cos B-\cos(A+B))}{\cos A\cos B\cos C}$$

$$=\frac{\sin C(\cos A\cos B-\cos A\cos B+\sin A\sin B)}{\cos A\cos B\cos C}$$

$$=\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C}=\tan A\tan B\tan C$$

81. Jika A, B, C sudut-sudut pada segitiga ABC, buktikan bahwa $\tan A \tan B \tan C \geq 3\sqrt{3}$!

Bukti :

$$a + b + c \geq 3\sqrt[3]{abc}$$
$$\tan A + \tan B + \tan C \geq 3\sqrt[3]{\tan A \tan B \tan C}$$
$$\tan A \tan B \tan C \geq 3\sqrt[3]{\tan A \tan B \tan C}$$
$$(\tan A \tan B \tan C)^3 \geq 27 \tan A \tan B \tan C$$
$$(\tan A \tan B \tan C)^2 \geq 27$$
$$\tan A \tan B \tan C \geq 3\sqrt{3}$$

82. Dalam segitiga ABC, buktikan bahwa $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$!

Bukti :

$$\sin A + \sin B + \sin C = 2\sin\left(\tfrac{1}{2}A + \tfrac{1}{2}B\right)\cos\left(\tfrac{1}{2}A - \tfrac{1}{2}B\right) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$
$$= 2\sin\tfrac{1}{2}(180^\circ - C)(\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B + \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$
$$= 2\cos\tfrac{1}{2}C(\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B + \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}C\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}C(\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C)$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}C(\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B + \sin\tfrac{1}{2}(180^\circ - (A + B))$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}C(\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B + \cos(\tfrac{1}{2}A + \tfrac{1}{2}B))$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}C(\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B + \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B)$$
$$= 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$
$$= 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

83. Jika A, B, C sudut-sudut pada segitiga ABC, buktikan bahwa

$$\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C \geq \tfrac{9}{4}\sec\tfrac{1}{2}A\sec\tfrac{1}{2}B\sec\tfrac{1}{2}C$$

Bukti :

Arithmetic Mean ≥ Harmonik Mean

$$\frac{a + b + c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \Leftrightarrow (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$
$$(\sin A + \sin B + \sin C)(\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C) \geq 9$$
$$\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C \geq \frac{9}{\sin A + \sin B + \sin C}$$
$$\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C \geq \frac{9}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$
$$\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C \geq \tfrac{9}{4}\sec\tfrac{1}{2}A\sec\tfrac{1}{2}B\sec\tfrac{1}{2}C$$

84. Buktikan bahwa $(a + b - c)(b + c - a)(c + a - b) \leq abc$

Bukti :

$$Karena\ (b - c)^2 \geq 0\ maka\ a^2 - (b - c)^2 \leq a^2$$
$$Karena\ (c - a)^2 \geq 0\ maka\ b^2 - (c - a)^2 \leq b^2$$
$$Karena\ (a - b)^2 \geq 0\ maka\ c^2 - (a - b)^2 \leq c^2$$
$$\left(a^2 - (b - c)^2\right)\left(b^2 - (c - a)^2\right)\left(c^2 - (a - b)^2\right) \leq a^2b^2c^2$$
$$(a + b - c)(a - b + c)(b + c - a)(b - c + a)(c + a - b)(c - a + b) \leq (abc)^2$$
$$(a + b - c)(b + c - a)(c + a - b) \leq abc$$

85. Jika a, b, c sisi-sisi segitiga ABC, buktikan bahwa $(a+b+c)(ab+bc+ca) \geq 9abc$!

<u>Bukti</u> :

$(a+b+c)(ab+bc+ca) = a^2b + abc + a^2c + ab_2 + b^2c + abc + abc + bc^2 + ac^2$

$= 3abc + (a^2b + bc^2) + (b^2c + a^2c) + (ac^2 + ab^2)$

$\geq 3abc + 2\sqrt{a^2b.bc^2} + 2\sqrt{b^2c.a^2c} + 2\sqrt{ac^2.ab^2} = 3abc + 2(abc + abc + abc) = 9abc$

86. Jika x bilangan real positif, buktikan bahwa $x^{2003} + x^{2001} + x^{1999} + ...+ \frac{1}{x^{1999}} + \frac{1}{x^{2001}} + \frac{1}{x^{2003}} \geq 2004$

<u>Bukti</u> :

$x^{2003} + \frac{1}{x^{2003}} \geq 2\sqrt{x^{2003}.\frac{1}{x^{2003}}} = 2$

$\left(x^{2003} + \frac{1}{x^{2003}}\right) + \left(x^{2001} + \frac{1}{x^{2001}}\right) + ...+ \left(x^1 + \frac{1}{x^1}\right) \geq 2 + 2 + ...+ 2 = 2.1002 = 2004$

87. Dalam segitiga ABC jika $a^2 = b^2 + c^2 + bc$ maka tentukan $\beta + \gamma$!

<u>Jawab</u> :

$a^2 = b^2 + c^2 + bc = b^2 + c^2 - 2bc\cos\alpha \quad \Rightarrow \quad \cos\alpha = -\frac{1}{2} \quad \Leftrightarrow \quad \alpha = 120^\circ$

$\beta + \gamma = 180^\circ - \alpha = 60^\circ$

88. Dalam segitiga ABC berlaku $c^2 = (a\cos\alpha - b\sin\alpha)^2 + (a\sin\alpha + b\cos\alpha)^2$. Tentukan besarnya sudut C !

<u>Jawab</u> :

$c^2 = a^2\cos^2\alpha - 2ab\sin\alpha\cos\alpha + b^2\sin^2\alpha + a^2\sin^2\alpha + 2ab\sin\alpha\cos\alpha + b^2\cos^2\alpha$

$c^2 = a^2(\cos^2\alpha + \sin^2\alpha) + b^2(\cos^2\alpha + \sin^2\alpha)$

$c^2 = a^2 + b^2 \quad \Rightarrow \quad \angle C = 90^\circ$

89. Tentukan nilai $\sin^2 15^\circ + \sin^2 15^\circ \cos^2 15^\circ + \sin^2 15^\circ \cos^4 15^\circ + \sin^2 15^\circ \cos^6 15^\circ + ...$

<u>Jawab</u> :

$\sin^2 15^\circ(1 + \cos^2 15^\circ + \cos^4 15^\circ + \cos^6 15^\circ + ...) = \sin^2 15^\circ.\frac{1}{1-\cos^2 15^\circ} = 1$

90. Sebuah balok luas alasnya 96 cm^2, luas sisi depannya 72 cm^2 dan luas sisi sampingnya 48 cm^2. Tentukan volume balok !

<u>Jawab</u> :

pl = 96 dan pt = 72 maka $l = \frac{96t}{72}$

lt = 48 atau $\frac{96t}{72}.t = 48 \quad \Rightarrow \quad t = 6 \quad \Rightarrow \quad l = 8 \; dan \; p = 12$

V = p l t = 12.8.6=576

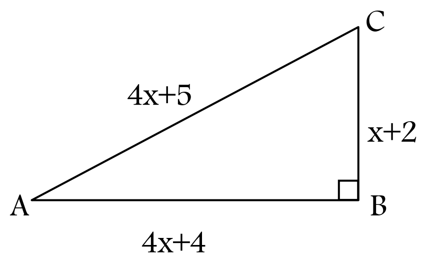
91. Sebuah balok mempunyai perbandingan ukuran p : l : t = 6 : 3 : 2. Jika panjang diagonal ruangnya 28 cm, tentukan volume balok !

<u>Jawab</u> :

Misal p = 6x, l = 3x dan l = 2x maka $28^2 = (6x)^2 + (3x)^2 + (2x)^2 \quad \Rightarrow \quad x = 4$

Jadi V = p l t = 24.12.8 = 2304 cm^2

92.



Tentukan luas segitiga ABC !

Jawab :

$$(4x+5)^2 = (4x+4)^2 + (x+2)^2 \quad \Rightarrow \quad x = 5$$

$$L = \frac{1}{2}(4.5+4)(5+2) = 84$$

93. Tentukan jumlah angka-angka dari $10^{25} - 25$!

Jawab :

$$10^{25} - 25 = \underbrace{1000.....000}_{25\text{ angka}} - 25 = \underbrace{999....999}_{23\text{ angka}}75$$

Jadi jumlah angka-angkanya = 9.23+7+5= 219

94. Jika f(1) = 5 dan f(x+1) = 2f(x) maka tentukan f(7) !

Jawab :

f(1) = 5

f(2) = 2f(1) = 10

f(3) = 2f(2) = 20

f(4) = 2f(3) = 40

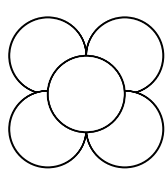
….

Jadi 5, 10, 20, 40, ….. Berupa barisan geometri dengan rasio = 2.

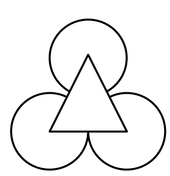
Sehingga f(7) = $5.2^{7-1} = 320$

95. Empat bola berjari-jari sama yaitu 10 cm terletak di atas meja sedemikian sehingga pusat dari keempat bola membentuk bujursangkar bersisi 20 cm. Bola kelima berjari-jari 10 cm diletakkan diatasnya sehingga bola tersebut menyinggung keempat bola pertama. Tinggi pusat bola kelima dari meja adalah ….

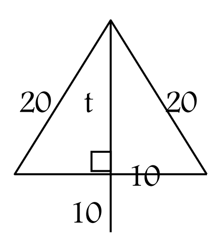
Jawab :



Dari atas



dari samping



h = t + 10 = $\sqrt{400 - 100} + 10 = 10\left(\sqrt{3} + 1\right)$

96. Pada lomba maraton tiap peserta diberi nomor urut 1, 2, 3, 4, ……dst. Banyaknya angka yang dipakai untuk menulis nomor seluruh peserta adalah 1998 buah. Berapa banyak peserta maraton tersebut ?

Jawab :

Nomor 1 angka : 9 x 1 = 9 yaitu 1 – 9

Nomor 2 angka : 90 x 2 = 180 yaitu 10 – 99

Nomor 3 angka : 1998 – 189 = 1809

Banyaknya bilangan dengan 3 angka : 1809 : 3 = 603

Jadi banyak peserta = 603 + 9 + 90 = 702 peserta